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Top Angle and the Maximum Speed of Falling Cones

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ABSTRACT *In the framework of a curriculum reform in the Netherlands, an interdisciplinary open inquiry assignment has been introduced as a part of the final secondary school examinations. We developed and evaluated a preparatory open inquiry assignment to be used in science and mathematics departments of schools, in collaboration with each other: the falling cones assignment. However, teachers encountered two content problems, that hindered the (guiding of) the open inquiry process. Physics educators solved the problems and showed how the solution can be used in guiding students for inquiry learning.*

KEY WORDS: *Inquiry learning, physics, teacher education.*

Introduction

Traditionally, laboratory activities in the classroom are based on a cookbook approach, hindering students to develop reflective thoughts on what they have done. When students are doing practical work, teachers rarely ask students if they understand what they are doing, why they are doing it, or what the results will show (Gallagher & Tobin 1987). Moreover, the teachers tend to pay much more attention to laboratory reports than to the process of inquiry and interpretation data. In a critical review, Hodson (1993) pointed out that practical work is often not taught very effectively, and, even in laboratory settings, only few students have the opportunity to develop an insight in doing investigations. In a recent curriculum reform for upper secondary education in the Netherlands, a shift towards doing investigations is promoted by introducing a *final open inquiry assignment* as a part of the school examination. Preparing for that kind of assignments should cross the boundaries of the single subjects (Millar, Lubben, Gott, & Duggan, 1994). Therefore, we developed a preparatory open inquiry assignment for the combination of physics and mathematics: *Falling Cones.*

The goal was to have students experience the research process from start to finish. Teachers were provided with draft teaching materials and adapted them to their school context. Using the materials in the classrooms, the students experienced two problems that could not be solved by their teachers, because of the complexity of physics. This paper describes the solution of the problems and the ways in which the solution can be used in guiding the students.

The Falling Cones Assignment

The *falling cones* open inquiry assignment is about studying the movement of a paper cone released at a certain height. The assignment is introduced by showing the movement of one cone first. Then, the movement of pairs of cones is demonstrated, and, among others, two cones of different sizes, having the same top angle and made of the same paper, as indicated in Figure 1, are used. These two cones are released at the same height and at the same time. To students' surprise, the cones fall at the same speed all the way. This brings them into discussions about what characteristics of a cone determine its movement.

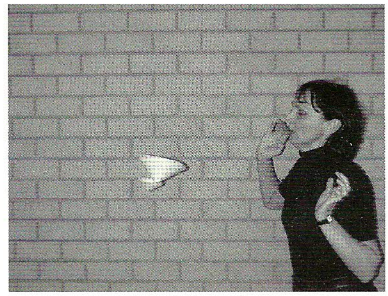

Figure 1. Paper Cones Falling Down

After this plenary introduction, the students in groups are challenged to make some interesting cones, and study the movements, as a 'pilot experiment' for a more extensive and focused investigation. Moreover, they study theory of gravitation and air resistance. They are asked to formulate a research question about paper cones, and to elaborate and implement a research plan. After having done the experiments and processed the data, the students formulate their conclusions. At the end, the groups present their results on a poster.

In Figure 2, the cone is drawn and relevant variables are shown. Using physics and mathematics theory, students can understand how the maximum speed (v_{max}) of the cone is related to variables such as its mass (m), the radius (r) of the ground circle, the air density (ϱ) and the drag coefficient (C):

$$F_{grav} = F_{drag} \qquad (1)$$

$$m \cdot g = \tfrac{1}{2} C A \varrho v^2_{max} \qquad (2)$$

$$v^2_{max} = \frac{mg}{\pi r^2} \cdot \frac{2}{C\varrho} \qquad (3)$$

Using these formulae and the drawing of Figure 2, students can make a design for an experimental or theoretical research project. The teacher guides the student groups when they are trying to formulate a research question, make a research design, do experiments or mathematical activities. In order to learn how to conduct a research project, teachers have to give students many opportunities to determine on their own what to investigate, how, and why to do it in a certain way.

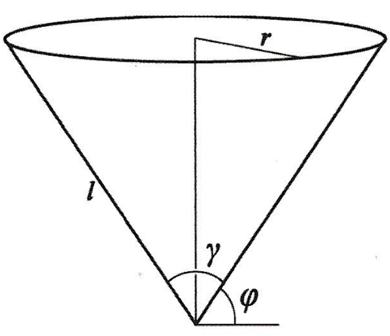

Figure 2. Scheme of a Cone with Variables

During their investigations, the students are guided by their science and mathematics teachers cooperating with each other. They help the students orient on the experimental as well as on theoretical aspects of the movement under study, and stimulate them to focus on aspects of the movement and the cone geometry during their investigation. No complete answers should be given, as it would make students' own contributions unnecessary. Instead, the teacher can ask for reasons and explanations, can answer questions by asking questions in return, can offer suggestions, etc., in order to stimulate students to make concrete what they want to know and investigate. To provide adequate guidance, teachers have to have a thorough understanding of the cone issue and how to investigate (Tamir, 1989).

The Issues of This Study

The falling cones assignment was done in four classes (grade 11 of the pre-university stream). The materials for doing the experiments were simple: rulers, stopwatches, scales, etc. No advanced equipment, such as a position sensor, was available. Experiences have shown that students were challenged to ask questions about the motion, about variables being relevant for the maximum speed, to probe relationships, and to do experiments. We learnt that some students asked questions that the teachers could not answer, but only by rules of thumb. The students appeared not to be content with those answers. We identified two content problems. The solving demands of these problems went beyond teachers' capabilities. It was our task as developers to solve them.

First Problem

Experiments about how the maximum speed of the cone depends on the mass, the shape of the cone, its radius etc., appeared to be most common with students. However, trying to measure the maximum speed, students encountered a problem: *what distance does the cone have to fall for its speed to become constant?*

From equation (3), it can be seen that one needs to have the value of the drag coefficient *C* for calculating that distance. The drag value depends on the shape of the cone, but no formula is available. So, teachers suggested to make a reasonable guess or said: just assume that after falling one meter, the speed is at its maximum. The students accepted these suggestions, though reluctantly. One group argued rightly: *The start phase of the fall may be dependent on the mass of the cone.* In a drawing (see Figure 3), they showed what they meant by 'start phase.'

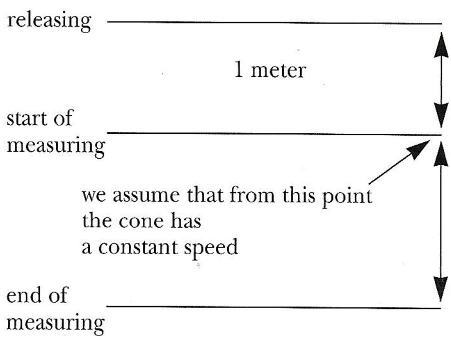

Figure 3. Transcription of the Group's Drawing Quoted in the Text, Where One Meter Is Used for "Start Phase."

Second Problem

Some students studied how the maximum speed depends on the top angle (γ) by releasing cones with equal mass and radius, but different top angles. They found that the bigger the top angle is, the slower the maximum speed. For getting a formula for this relation, they studied the theory and concluded: *If you insert all known values into formula (3), the drag coefficient and the speed are left as variables.* They measured v_{max} for three cones with different top angles and wanted to make a graph of v_{max} and something with γ (e.g., sin γ) to try to make a formula, but did not succeed. They asked their teacher which formula to try. He suggested $v^2_{max} \propto (\sin \gamma)^{-1}$, but in fact had no basis for it. So, the problem is: What does the relation between v_{max} and sin γ look like?

First, an answer will be given to the question: How does the drag value depend on the top angle? This is done in two steps: (a) by developing a model, and (b) by doing experiments and evaluating results using the model. Next, the solution of the two problems and consequences for guiding the students will be discussed.

Developing a Model for the Cone

The drag coefficient C is dependent on the top angle of a cone. This can be illustrated by looking at the drag coefficients of some shapes comparable to cones, which are found in the literature, as it is shown in Table 1. The flat circle can be regarded as a cone with a top angle of 180°. Cones with sharper top angles will have smaller drag coefficients than the flat circle, but no less than the drag coefficient of the open half sphere. For, it has an open base like the cone and an "ideal" water-drop-like top. So, the cone drag coefficients are expected to have values between 0,34 and 1,1.

Table 1.
Drag Coefficient of Some Different Shapes (Vademecum, 1995)

Flat circle	1.11
Open half sphere	0.34
Drop of water	0.06

Newton already studied the resistance objects experience when falling through a homogeneous medium. Edwards (1997) presented a simplified version of the complicated Principia Mathematica theory. His version was used to find a relationship between the cone drag coefficient and the top angle. Newton assumed that the air consists of tiny elastic particles with mass m, uniformly distributed in space, and having no speed. If the cone is taken as the reference system, the air particles are moving vertically upward with speed v, as indicated in Figure 4. As the directions of the

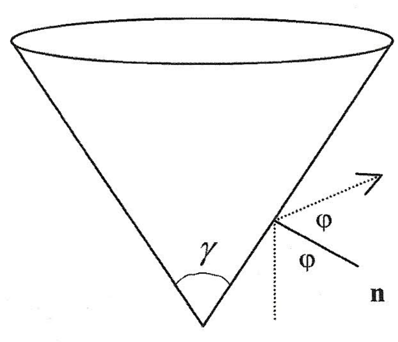

Figure 4. A Particle Colliding Elastically with the Cone

velocities of the actual air molecules are uniformly distributed in space, this model can be applied here as a first order approximation. It has to be noted that collisions between particles are not accounted for. The bouncing against the exterior causes a drag force. Using the model, the following formula for the drag force can be derived, as it is explained in Appendix A:

$$F_{drag} = 2A\varrho v^2 \cdot \sin^2(\gamma/2) \qquad (4)$$

Using formulae (4) and (2), one finds for the drag coefficient:

$$C(\gamma) = 4\sin 2(\gamma/2) \qquad (5)$$

This result predicts that $C = 4$ when the top angle is 180°, and it tends to 0 when the top angle approaches 0°. Following this first model, $0 < C \leq 4$, this does not agree with our expectation $0{,}34 < C < 1{,}1$. So, the Newton model is not adequate. This is because collisions between particles are not included in the model, resulting in airstreams. At the open side of the falling cone, turbulence effects occur, as is the case with the open half sphere. So, we guess that a turbulence factor a, not depending on γ, has to be added to the theoretical drag value, its magnitude probably similar to that of C of the half sphere (~0.34). At the bottom, laminar airstreams may play a role, not affecting the $\sin^2(\gamma/2)$ dependence, but lowering its coefficient below 4. So, formula (5) has to be changed into:

$$C(\gamma) = a + b \,.\, \sin^2(\gamma/2) \qquad (6)$$

If γ = 180° (flat circle), the drag coefficient is 1,1. So, $a + b$ should be about 1,1.

The Experiments

To study the relation between the drag coefficient and the top angle, a series of cones was constructed. All had the same base area A, but different top angles γ. To keep A constant, the radius r of the base has to be kept constant. Therefore, an increase of top angle γ means a decrease of the length l of the cone. The mass of each cone was made equal by adding some weight.

A position sensor (CBR of Texas Instruments) was used to measure the position of the falling cones at different moments of time. Using an interface (Coachlab II, CMA 2000), the sensor signal was sent to the computer and processed by the program Coach 5 to produce distance-time and speed-time graphs. From the graphs, the maximum speed of every cone was calculated, and the average of three measurements was used in the calculations.

Formula (3) was used to calculate the drag coefficients. Results were plotted in a C, $\sin^2(\gamma/2)$ diagram.

When doing the measurements, some difficulties were encountered. The cone tended to deviate, when released less than 1 meter from the wall of the laboratory room. The height of the room (4,5 m) appeared to be too small for the sharper cones to leave a sufficient distance for measuring the maximum speed. Therefore, no reliable data could be gathered for cones with a top angle less than 60°. For cones with $\gamma > 120°$, it was not possible to select three movements without fluttering. Eventually, the data of seven cones with top angles between 60° and 120° could be used.

Results

In Figure 5, a typical diagram produced by Coach5 is shown. It illustrates that, some time after release, the (x, t) graph shows a straight line, the (v, t) graph reaching a maximum.

The drag coefficient is plotted versus $\sin^2(\gamma/2)$ in Figure 7. This diagram confirms formula (6), the coefficients being

$a = 0{,}40 \pm 0{,}04$

$b = 0{,}57 \pm 0{,}11$

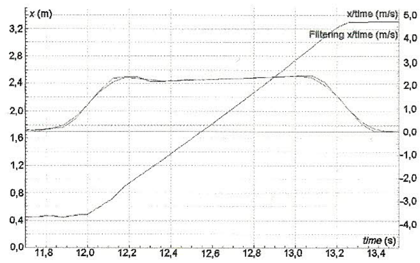

Figure 5. A Diagram from Coach5

Discussion

The linear relation between the drag value and $\sin^2(\gamma/2)$ is confirmed within the realm of $60^o < \gamma < 120^o$. Extrapolation to $\gamma = 0$ gives: $C = 0{,}40 \pm 0{,}04$. Extrapolation of the relation to $\gamma = 180^o$ gives $C = 0{,}97 \pm 0{,}15$. These values agree with the expected order (0,34 and 1,1 respectively).

Consequences: Make a Reasonable Guess

Having found the formula (6) and the values of its coefficients, we developed ways in which students can be guided. The first problem was: *What distance does the cone have to fall for its speed to become constant?*

To solve this, Polya's approach of making a first and a second 'reasonable guess' can be used (Polya, 1954). In the first 'reasonable guess,' students are asked to use prior knowledge: the distance a free falling body has to fall to reach the maximum speed is given by the 'uniform acceleration' formula:

$$s = \frac{v^2_{max}}{2g} \qquad (7)$$

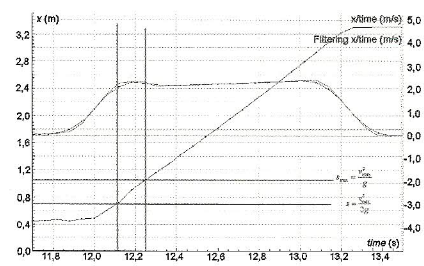

Figure 6. Estimate of Distance of Falling before Reaching Maximum Speed

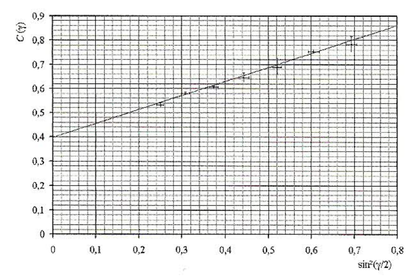

Figure 6. Diagram with the Drag Coefficient as a Function of the Square of the Sine of the Top Angle.

Using our experimental results and inserting the maximum speed from the Figure 4 graphs into (7), we get as first *guess* = 0,3 m. This is represented by the lower horizontal line in Figure 6. The crossing point of this line with the (x, t) graph is not on the linear part of the graph. So, the cone is not yet at its maximum speed after having fallen 0,3 m. So, a second guess is needed. The upper line in Figure 6 at double distance does cross the linear part. The same results were found when we applied this second guess to the other cones we used. So, an easy way can be chosen: double the distance! The teacher can suggest the students to solve the first problem by taking the distance two or three times the result of formula (7). They need a further suggestion, because they are likely to say they cannot use that formula without knowing the maximum speed. Then, it could be suggested to make a reasonable guess from what point onwards the speed is constant, to measure the approximated maximum speed. And then check it using formula 7.

An alternative for solving problem 2 is modelling with the computer (Mooldijk & Savelsbergh, 2000). It takes a lot of time and could be a separate research project, but there is room for students to find their own ways!

The second issue was: *How does the maximum speed depend on the top angle?* From a guidance point of view, it is better to transform it into: What graph is the best to make, to process the data gathered when studying how the maximum speed depends on the top angle? The teacher can suggest the following procedure to the students:

- calculate the drag coefficient of all cones from experimental data, using formula 3.
- try a graph of the drag coefficient versus $\sin(\gamma/2)$ or $\sin^2(\gamma/2)$.

When trying to measure the maximum speed, they will meet problem 1. As it can be solved now without using formula 6, they can find the formula themselves!

Conclusions

In this study, the relation between the drag coefficient and the top angle of a cone can be found using formula 6. Ways of using this formula to answer the questions that students may ask are suggested, using formula 7. Three important, more general aspects of our suggestions have to be stressed.

(1) The Polya method: Make a reasonable guess using what you already know; check the result and, if needed, adapt your guess. This method is an important one in the light of the main goal of the investigation assignments: Learning about doing investigations.

(2) The different roles of the teacher and of the physics curriculum expert. The teacher has to concentrate on the actual guiding of the students. He/she therefore needs the expert to identify physics content problems, which hinder the guidance and to do the deep-going and time-consuming development work to find handsome solutions.

(3) Knowledge of a complicated formula, like formula 6, can be used by teachers to give direction to the investigation process of students, but should not be given to the students. Otherwise, the room for them to find their own solutions would be too small.

Appendix

In this Appendix formula (4) is derived using the Newton model.

Let N be the number of particles per unit of volume. All particles hit the cone surface at angle φ with the surface normal, and bounce off with the same angle. From Figure 2 and 4, it can be seen that φ equals $\pi/2 - \gamma/2$. If a particle with mass collides elastically with the cone, its momentum p changes by:

$$\Delta p_{particle} = 2mv \,.\, \cos(\varphi) \qquad (8)$$

along the normal on the outside cone surface. This results in the opposite change of momentum of the cone:

$$\Delta p_{cone} = 2mv \,.\, \cos(\varphi) \qquad (9)$$

As the horizontal components of the momentum of the particles that bounce on the cone are cancelled out, only the vertical components count. So, one collision contributes just a change of momentum of the cone in the vertically upward direction of:

$$\Delta p_{vertical} = 2mv \,.\, \cos^2(\varphi) \qquad (10)$$

The number of particles that strike an area dA of the cone surface in time Δt is

$$N_{dA} = Nv \,.\, \cos(\varphi)\, dA \,.\, \Delta t \qquad (11)$$

Using the equality of change of impulse and momentum, ϱ being the density of the medium the cone is falling in, with

$$\varrho = Nm \qquad (12)$$

the vertical force on the surface of an area dA from air drag is

$$dF_{drag} = 2\varrho v^2 \,.\, \cos^3(\varphi) \,.\, dA \qquad (13)$$

The air drag on the total cone surface S is given by the surface integral:

$$F_{drag} = \iint_S 2\varrho v^2 \,.\, \cos^3(\varphi) \,.\, dA \qquad (14)$$

This equation can be written as follows:

$$F_{drag} = 2\varrho v^2 R \qquad (15)$$

R equalling:

$$R = \iint_S \cos^3(\varphi) \,.\, dA \qquad (16)$$

According to Edwards (1977), one can write for the coefficient of resistance:

$$\frac{R}{\pi} = \int_0^r \frac{2x}{1+y'(x)^2} \,.\, dx \qquad (17)$$

with $y(x)$ the function that determines the shape of the falling body. In the case of the cone, you can write for that function

$$y(x) = ax \qquad (18)$$

a being

$$a = \tan\varphi = \frac{\sin\varphi}{\cos\varphi} \qquad (19)$$

So $y' = a$.

Substitution of the requirement (19) for the cone into (18) gives:

$$\frac{R}{\pi} = \int_0^r \frac{2x}{1+y'(x)^2} \,.\, dx = \int_0^r \frac{2x}{1+a^2} \,.\, dx = \frac{r^2}{1+a^2} \qquad (20)$$

Inserting (20) into (21), using $\varphi = {}^{\pi}/_{2} - {}^{\gamma}/_{2}$ and $A = \pi\ r^2$ one gets:

$$R = A \,.\, \cos^2 (\pi/2 - \gamma/2) = A \,.\, \sin^2 (\gamma/2) \qquad (21)$$

So, we can write the drag force:

$$F_{drag} = 2A\varrho v^2 \,.\, \sin^2 (\gamma/2) \qquad (22)$$

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